UDC 681.518

SOLUTION OF THE PROBLEM OF COMPUTATIONAL STABILITY AND CONSISTENCY OF SAMPLE ESTIMATES OF THE CORRELATION MATRIX OF OBSERVATIONS BY THE METHOD OF DYNAMIC REGULARIZATION

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ВИРІШЕННЯ ПРОБЛЕМИ ОБЧИСЛЮВАЛЬНОЇ СТІЙКОСТІ І СПРОМОЖНІСТЬ ВИБІРКОВИХ ОЦІНОК КОРЕЛЯЦІЙНОЇ МАТРИЦІ СПОСТЕРЕЖЕНЬ МЕТОДОМ ДИНАМІЧНОЇ РЕГУЛЯРИЗАЦІЇ

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Annotation. *The problem of forming sample estimates of the correlation matrix of observations that satisfy the criterion "computational stability – consistency" is considered. The variants in which the direct and inverse asymptotic forms of the correlation matrix of observations are approximated by various types of estimates formed from a sample of a fixed volume are investigated. The consistency of computationally stable estimates of the correlation matrix for their static regularization was analyzed. The contradiction inherent in the problem of regularization of the estimates with a fixed parameter is revealed. The dynamic regularization method as an alternative approach is proposed, which is based on the uniqueness theorem for solving the inverse problem with perturbed initial data. An optimal mean-square approximation algorithm has been developed for dynamic regularization of sample estimates of the correlation matrix of observations, using the law of monotonic decrease in the regularizing parameter with increasing sample size. An optimal dynamic regularization function was obtained for sample estimates of the correlation matrix under conditions of a priori uncertainty with respect to their spectral composition. The preference of this approach to the regularization of sample estimates of the correlation matrix under conditions of a priori uncertainty is proved, which allows to exclude the domain of computational instability from solving the inverse problem and obtain its solution in real time without involving prediction data and additional computational cost for finding the optimal value of the regularization parameter. The application of the dynamic regularization method is shown for solving the problem of detecting a signal at the output of an adaptive antenna array in a nondeterministic clutter and jamming environment. The results of a computational experiment that confirm the main conclusions are presented.*

Keywords: sample estimates*, correlation matrix, dynamic regularization, probability convergence of estimate, computational stability of estimate, consistency of estimate*

Анотація. Розглянуто проблему формування вибіркових оцінок кореляційної матриці спостережень, які задовольняють критерій «обчислювальна стійкість - спроможність». Досліджено ситуації, коли пряма і зворотні асимптотичні форми кореляційної матриці спостережень апроксимуються оцінками різного виду, які сформовані за вибіркою фіксованого обсягу. Піддана аналізу спроможність стійких в обчислювальному відношенні оцінок кореляційної матриці за їх статичної регуляризації. Виявлено протиріччя, властиве задачі регуляризації цих оцінок за фіксованого параметру. Запропоновано альтернативний підхід – метод динамічної регуляризації, який спирається на теорему єдиності розв'язку оберненої задачі зі збуреними вихідними даними. Розроблено оптимальний за середньоквадратичним наближенням алгоритм динамічної регуляризації вибіркових оцінок кореляційної матриці спостережень, який використовує закон монотонного убування регуляризованого параметру за умови зростання обсягу вибірки. Отримано оптимальну функцію динамічної регуляризації вибіркових оцінок кореляційної матриці в умовах апріорної невизначеності щодо їх спектрального складу. Визначено переваги оптимальної динамічної регуляризації в сенсі обчислювальної стійкості та спроможності вибіркових оцінок кореляційної матриці спостережень. Доведено перевагу зазначеного підходу до регуляризації вибіркових оцінок кореляційної матриці в умовах апріорної невизначеності, що дозволяє виключити з розв'язання оберненої задачі область обчислювальної нестійкості та отримати її рішення в режимі реального часу без залучення даних прогнозування і додаткових обчислювальних витрат на пошук оптимального значення параметра регуляризації. Показано докладання методу динамічної регуляризації до вирішення задачі виявлення корисного сигналу на виході адаптивної антенної решітки у недетермінованій завадовій ситуації. Представлені результати обчислювального експерименту, які підтверджують основні висновки.

Ключові слова: саморегуляризація, статична регуляризація, динамічна регуляризація, збіжність оцінки (за ймовірністю), критерій спроможності оцінок, кореляційна матриця.

FORMULATION OF THE PROBLEM. ANALYSIS OF STUDIES AND PUBLICATIONS

Inversion of the correlation matrix of observations belongs to the class of problems associated with the reversal of cause-effect relationships. This procedure is the basis for solving inverse statistical problems in applications of spectral analysis, space-time processing of multidimensional signals, control theory, identification, forecasting and decision making [1-6].

Practical solution of such problems involves the replacement of the asymptotic form of the N-dimensional correlation matrix by its sample estimate, formed on a finite time interval $[0,T]$ in L iterations using known computational algorithms [5-9]. These algorithms tend to monotonically increase the rank of the evaluation matrix to the full value when $L = N$. Due to the inevitability of the situation when $L < N$, the indicated evolution in real time raises the problem of the degeneration of the observation matrix. This leads to the loss of computational stability of inverse problems on an indefinite interval of iterations $L \in [1, N-1]$, when the dimension of the system N has an arbitrary value.

The concept of overcoming the indicated problem is connected with the use of regularization methods [5-8], which make it possible to obtain computationally stable estimates of the correlation matrices synchronously with the development of the observed process. The basis of these methods is the search for regularizing operators, in which the rule for choosing the regularization parameter μ takes priority.

In the classical formulation, the problem of finding the optimal value of the static regularization parameter μ goes back to the work of Tikhonov A.N. [10] and is solved by residual, trial-and-error or iterative regularization methods. In particular, when solving a perturbed system of linear algebraic equations, the search for the regularization parameter μ is organized in such a way that the residual of the approximate solution is comparable in magnitude with the level of accuracy of the initial data of the inverse problem [3, 10, 11]. The proper search for the regularization parameter μ is carried out on a certain set of values, and the choice of the optimal parameter μ is based on a priori information [10]. However, in solving the inverse problem in real time, these methods are characterized by resource constraints of a computational nature, as well as the need for additional a priori information about the structure of the solution of the optimization problem and the level of errors of the initial data.

Methods for solving inverse problems with regularization of the maximum likelihood (ML) estimate of the correlation matrix of observations are considered in a number of works [3, 6-8]. They are classified as methods of static regularization [3, 10], when the problem of zero eigenvalues is solved by shifting the spectrum of the estimate of the correlation matrix to the right by some constant number μ. In this case, the regularized matrix has similar, but not identical, properties of the initial estimate in the sense of its consistency. Methods of regularization of the sample estimate of the correlation matrix are characterized by both a limitation of the computational resource and specific informational limitations. In particular, the determination of the optimal value of the regularization parameter requires information about the true or expected interference-to-noise ratio, the spectral composition of the correlation observation matrix and the possible number of noise sources [7, 8]. In a non-deterministic situation, getting such information is very problematic.

These restrictions give rise to a dialectical contradiction according to the criterion "computational stability – consistency" of the estimate [10, 11]. Indeed, the value of the regularization parameter μ should be, on the one hand, sizeable, which guarantees the computational stability of the inverse problem, and, on the other hand, small, in order to influence the matrix estimate in the sense of its consistency in the sample size as little as possible. Overcoming the existing contradiction actualizes the problem of studying the dynamic regularization of a sample estimate of the correlation matrix in relation to solving inverse problems under conditions of a priori uncertainty.

The concept of regularization of inverse problems associated with the dynamics of the initial data is described by Osipov Yu.S. [12], where it is shown that the regularization parameter μ of the inverse problem should be updated in real time. This judgment to some extent has found its development in a wide range of applications. In the work [5] the procedure for regularizing the ML estimate of the correlation matrix of interference based on the monotonic decrease of its diagonal complement with increasing training sample size is considered and the influence of the "weight" of the diagonal complement on the convergence process of the energy criterion for different estimates of the correlation matrix is shown. However, the problem of choosing the optimal "weight" of a diagonal complement on a set of possible values remained open. In this regard, the demand is to determine the optimal dynamic regularization function of the ML estimate of the correlation matrix of observations. Solving such a problem involves determining the dependence of the "weight" (constant) of the regularizing parameter on the size of the training sample. Research in this direction can complement the results associated with the problem of calculating stable and consistent estimates of correlation matrices.

The main contribution of this work is the application of the method of dynamic regularization of sample estimates of the correlation matrix of observations to satisfy the criterion "computational stability – consistency" in real time. In the general methodological context, the indicated problem is solved on a multidimensional complex Gaussian distribution. The known properties of such a distribution make it possible to analyse the processes of convergence of sample estimates of correlation matrices in the sense of their computational stability and consistency under static and dynamic regularization.

The rest of this paper is organized as follows. Section 2 and 3 present the studies of computational stability and consistency of estimates of correlation matrices and the consistency of estimates of correlation matrices for static regularization. Section 4 present the method of dynamic regularization of sample estimates of the correlation matrix and the main result of this work – the algorithm for calculating the optimal parameter of the dynamic regularization. The effectiveness of the algorithm is illustrated via simulations for adaptive antenna array example in Section 5. Finally, Section 6 concludes this paper summarizing the main findings and giving some recommendations for future work.

STUDY OF COMPUTATIONAL STABILITY AND CONSISTENCY OF ESTIMATES OF CORRELATION MATRICES

Let the stationary random vector process $\mathbf{u}(t)=\mathbf{s}(t)+\mathbf{n}(t)$, which is an additive mixture of orthogonal vectors of the signal $\mathbf{s}(t)$ and noise $\mathbf{n}(t)$, is observed in the N-dimensional Hilbert space. The norms of the signal and noise vectors satisfy conditions $\|\mathbf{s}(t)\|<\infty$, $\|\mathbf{n}(t)\|<\infty$. The second-order statistical moments of vectors $\mathbf{s}(t)$ and $\mathbf{n}(t)$ with the δ-correlation of noise and infinity of the observation interval T are [3, 4]

$$\mathbf{A}_S=\lim_{T\to\infty}\left[T^{-1}\int_0^T \mathbf{s}(t)\mathbf{s}^H(t)\,\mathrm{d}t\right],\ \mathbf{A}_n=\lim_{T\to\infty}\left[T^{-1}\int_0^T \mathbf{n}(t)\,\mathbf{n}^H(t)\,\mathrm{d}t\right]=P_n\,\mathbf{I},\ \lim_{T\to\infty}\left[T^{-1}\int_0^T \mathbf{s}(t)\,\mathbf{n}^H(t)\,\mathrm{d}t\right]=\mathbf{0},$$

where P_n – noise power; ^{H} – denotes Hermitian (or complex-conjugate transpose) matrix operation; $\mathbf{0} = \mathbf{0}_{N\times N}$, $\mathbf{I} = \mathbf{I}_{N\times N}$ – zero and unit N-dimensional matrices.

Based on the adopted model, the asymptotic forms of the direct $\mathbf{A}$ and inverse $\mathbf{A}^{-1}$ correlation matrices of the observation vector $\mathbf{u}(t)$ are determined by the limits

$$\mathbf{A} = \lim_{T\to\infty}\left[\frac{1}{T}\int_0^T \mathbf{u}(t)\mathbf{u}^H(t)\,\mathrm{d}\,t\right] = \mathbf{A}_S + P_n\cdot\mathbf{I},$$

$$\mathbf{A}^{-1} = \left\{\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T \mathbf{u}(t)\mathbf{u}^H(t)\,\mathrm{d}\,t\right]\right\}^{-1} = (\mathbf{A}_S + P_n\cdot\mathbf{I})^{-1}.$$

The full rank of the correlation matrix $\mathbf{A}$: $\mathrm{rank}(\mathbf{A}) = N$ always guarantees the existence of the inverse matrix $\mathbf{A}^{-1}$. In practical applications with a finite interval of observations $[0,T]$ the asymptotic representations of matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ are replaced by their estimates (discrete analogues) $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}^{-1}$ [6-9]. Such estimates are calculated as source data become available and, under certain conditions, do not characterized by the full rank.

Let us project the problem of forming estimates of matrices $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}^{-1}$ on a grid of time samples, assuming that for a finite sample of size L from a set of vectors $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_L$, a multidimensional distribution density is given by

$$p(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_L) = (\pi^N \det\mathbf{A})^{-L}\exp\left\{-\sum_{k=1}^{L}\mathbf{u}_k^H\mathbf{A}^{-1}\mathbf{u}_k\right\}.$$

Then ML estimate $\tilde{\mathbf{A}}(L)$ of the matrix $\mathbf{A}\in\Omega_H$ can be written as follows [3, 4, 6-9]:

$$\tilde{\mathbf{A}}(L) = L^{-1}\sum_{k=1}^{L}\mathbf{u}_k\,\mathbf{u}_k^H, \tag{1}$$

where Ω_H – set of N-dimensional Hermitian matrices.

Algorithm (1) reflects the process of direct summation of single-rank matrices $\tilde{\mathbf{A}}_k = \mathbf{u}_k\,\mathbf{u}_k^H\ \forall k\in[1,L]$ in real time. With an increase in the sample size of L to the dimension of the matrix N, that is $L = N$, the estimate of $\tilde{\mathbf{A}}(L)$ reaches the full rank $\mathrm{rank}\left[\tilde{\mathbf{A}}(L)\right] = N$. A further increase in sample size $L: L > N$ in the presence of internal noise is accompanied by natural regularization (self-regularization) of the matrix $\tilde{\mathbf{A}}(L)$. In this case, the estimate of the correlation matrix $\tilde{\mathbf{A}}(L)$ and its inversion $\tilde{\mathbf{A}}^{-1}(L)$ tend to their asymptotic forms [9]: $\lim_{L\to\infty}\tilde{\mathbf{A}}(L) = \mathbf{A}$, $\lim_{L\to\infty}\tilde{\mathbf{A}}^{-1}(L) = \mathbf{A}^{-1}$. Equation (1) and its recurrent computational modification represented at the *k*-th step

$$\tilde{\mathbf{A}}_k = k^{-1}\left[(k-1)\,\tilde{\mathbf{A}}_{(k-1)} + \mathbf{u}_k\,\mathbf{u}_k^H\right],\quad k\in[1,L] \tag{2}$$

with the initial condition $\tilde{\mathbf{A}}_1 = \mathbf{u}_1\,\mathbf{u}_1^H$, allows to obtain the direct and recurrent forms of the estimate's inversion $\tilde{\mathbf{A}}^{-1}(L)$ for an arbitrary sample size L:

$$\tilde{\mathbf{A}}^{-1}(L) = L\left[\sum_{k=1}^{L}\mathbf{u}_k\,\mathbf{u}_k^H\right]^{-1}, \tag{3}$$

$$\tilde{\mathbf{A}}^{-1}(L) \equiv \tilde{\mathbf{A}}_L^{-1} = L\left[(L-1)\,\tilde{\mathbf{A}}_{(L-1)} + \mathbf{u}_L\,\mathbf{u}_L^H\right]^{-1}. \tag{4}$$

Decomposition of a recurrent computational scheme (4) in accordance with the rule [7] gives the following result

$$\tilde{\mathbf{A}}^{-1}(L) \equiv \tilde{\mathbf{A}}_L^{-1} = \frac{L}{(L-1)} \left\{ \mathbf{I} - \frac{\tilde{\mathbf{A}}_{(L-1)}^{-1} \ \mathbf{u}_L \mathbf{u}_L^H}{L + \mathrm{tr}\left[\tilde{\mathbf{A}}_{(L-1)}^{-1} \ \mathbf{u}_L \mathbf{u}_L^H \right]} \right\} \times \tilde{\mathbf{A}}_{(L-1)}^{-1}, \tag{5}$$

where $\mathrm{tr}\{\bullet\}$ – spur of matrix.

In computational practice, the criterion of stability and consistency of estimates (1)-(5) is the convergence of the corresponding matrix norms [11]:

$$\varepsilon(L) = \frac{\left\| \mathbf{A} - \tilde{\mathbf{A}}(L) \right\|^2}{\left\| \mathbf{A} \right\|^2}; \tag{6}$$

$$\beta(L) = \frac{\left\| \mathbf{A}^{-1} - \tilde{\mathbf{A}}^{-1}(L) \right\|^2}{\left\| \mathbf{A}^{-1} \right\|^2}. \tag{7}$$

The estimate is considered to be stable according to Hadamard, if for any sample size L the norm of approximation $\beta(L)$ has a final value of $\beta(L) = \zeta < \infty$, where ζ – some positive number.

Estimates are considered consistent if the property of their strong convergence to the corresponding asymptotic forms of matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ is satisfied: $P\left\{ \lim_{L\to\infty} \varepsilon(L) = 0 \right\} = 1$, $P\left\{ \lim_{L\to\infty} \beta(L) = 0 \right\} = 1$, here $P\{\bullet\}$ – the probability of the event $\{\bullet\}$.

Remark 1. The difficulty, and sometimes the impossibility, of obtaining analytical dependencies $\varepsilon(L)$ and $\beta(L)$ is caused by[11]:

− first, the uncertainty of the results due to the degeneration of estimates of the N-dimensional Hermitian matrix of incomplete rank $\tilde{\mathbf{A}}(L)$ in the area of loss of computational stability $G : G\{L : L < N, \beta(L) = \infty\}$;

− secondly, the complexity of describing the statistical distribution of eigenvalues and unitary vectors of a random matrix $\tilde{\mathbf{A}}(L)$ with an arbitrary sample size.

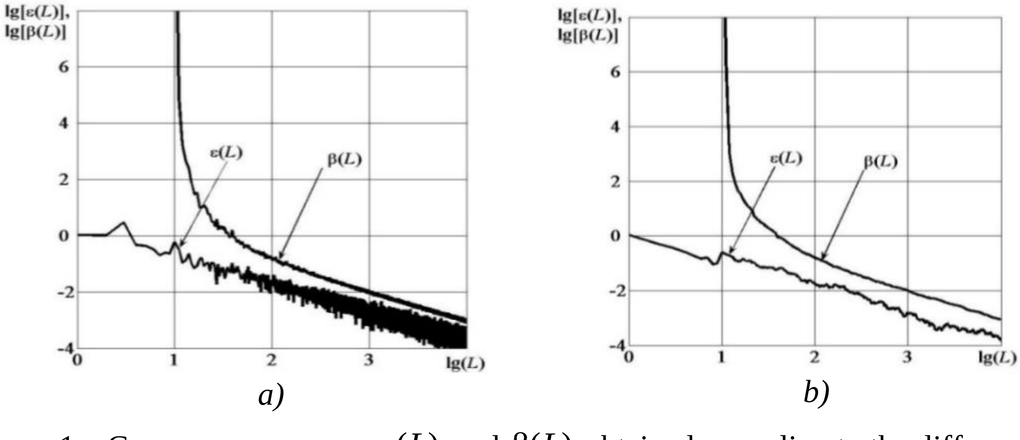


Figure 1 – Convergence curves $\varepsilon(L)$ and $\beta(L)$ obtained according to the different algorithms: (a) – algorithms (1) and (3); (b) – algorithms (2) and (4)

A natural way to overcome the above limitations is to conduct simulation studies that are reliable in the sense of convergence of estimates $\tilde{\mathbf{A}}(L)$, $\tilde{\mathbf{A}}^{-1}(L)$ to the corresponding asymptotic forms. The convergence of computational algorithms (1)-(5) demonstrate the simulation results, which are presented in Figures 1 and 2 in the form of averaged nonstationary dependences $\varepsilon(L)$ and $\beta(L)$ with $N = 10$.

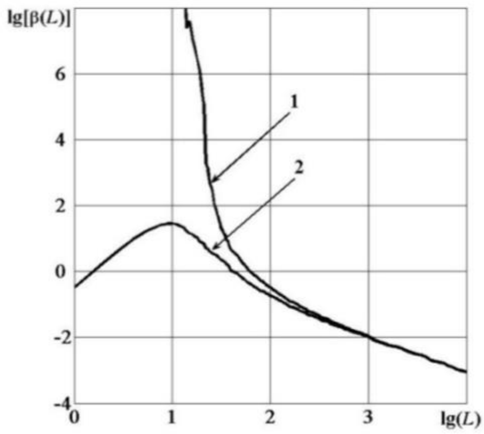


Figure 2 – Curves $\beta(L)$ obtained according to the algorithm (5): Case 1 – $\tilde{\mathbf{A}}_1 = \mathbf{u}_1 \mathbf{u}_1^H$;
Case 2 – $\tilde{\mathbf{A}}_1 = \mathbf{I}$

Figure 1, a shows the convergence curves $\varepsilon(L)$ and $\beta(L)$ of the estimates $\tilde{\mathbf{A}}(L)$ and $\tilde{\mathbf{A}}^{-1}(L)$ respectively, obtained in the process of direct summation of single-rank matrices (1) with the subsequent inversion of the result of the summation according to algorithm (3).

Figure 1, b shows convergence curves $\varepsilon(L)$ and $\beta(L)$ obtained in accordance with the procedure for the formation of estimates $\tilde{\mathbf{A}}(L)$ by the recurrent algorithm (2) with further inversion of the result of the recurrent approximation using algorithm (4).

Figure 2 shows the convergence $\beta(L)$ of estimate $\tilde{\mathbf{A}}^{-1}(L)$ obtained by the recurrent algorithm (5) for the following initial conditions: $\tilde{\mathbf{A}}_1 = \mathbf{u}_1 \mathbf{u}_1^H$ – curve 1 and $\tilde{\mathbf{A}}_1 = \mathbf{I}$ – curve 2.

The tendency of the behaviour of dependencies $\varepsilon(L)$ and $\beta(L)$ indicates a number of features inherent in algorithms for calculating estimates $\tilde{\mathbf{A}}(L)$ and $\tilde{\mathbf{A}}^{-1}(L)$:

- estimates (1)-(5) belong to the class of consistent estimates;
- algorithms (2), (4) and (5) due to their recurrent form have the property of smoothing estimates $\tilde{\mathbf{A}}(L)$ and $\tilde{\mathbf{A}}^{-1}(L)$, which indicates their effectiveness – the minimum variance of estimates with respect to direct summation algorithms (1) and (3);
- algorithms (3)-(5) are characterized by the objective existence of a region of loss of computational stability G of estimate $\tilde{\mathbf{A}}^{-1}(L)$ for the initial condition $\tilde{\mathbf{A}}_1 = \mathbf{u}_1 \mathbf{u}_1^H$ (in this case with $L < N = 10$). However, under condition $\tilde{\mathbf{A}}_1 = \mathbf{I}$, estimate (5) will be computationally stable, but it will not satisfy the approximation criterion in norm $\beta(L) < 1$ over the entire range of values L (Figure 2, curve 2).

Remark 2. Estimates (3)-(5) of matrix $\tilde{\mathbf{A}}^{-1}(L)$, despite their consistency in terms of sample size L, have an area of loss of computational stability G, in which, under constraint $L < N$, approximation $\beta(L) \to \infty$. As is known, computationally stable estimates of matrices can be obtained using static regularization method [6-8, 10]. In this case, it is appropriate to investigate the question of the consistency of such estimates.

STUDY OF THE CONSISTENCY OF ESTIMATES OF CORRELATION MATRICES FOR STATIC REGULARIZATION

Static regularization implies a "forced" shift of the spectrum of the initial estimate of matrix $\tilde{\mathbf{A}}(L)$ to the right by a fixed value of the regularizing parameter $\mu = f(\xi) > 0$, which is consistent with the level of error ξ [7, 8]:

$$\tilde{\mathbf{A}}_\mu(L) = \tilde{\mathbf{A}}(L) + \mu \mathbf{I}\,.$$

This guarantees the computational stability of the estimate $\tilde{\mathbf{A}}_\mu^{-1}(L) = \left[\tilde{\mathbf{A}}(L) + \mu \mathbf{I}\right]^{-1}$, regardless of the size of the sample L: $\left\| \tilde{\mathbf{A}}_\mu^{-1}(k) \right\| < \infty \forall k \in [1, L]$.

By analogy with (6) and (7), we investigate the consistency of estimates of the direct $\tilde{\mathbf{A}}_\mu(L)$ and inverse $\tilde{\mathbf{A}}_\mu^{-1}(L)$ matrices by the convergence criterion of regularized matrix norms:

$$\varepsilon_\mu(L) = \frac{\left\| \mathbf{A} - \tilde{\mathbf{A}}_\mu(L) \right\|^2}{\left\| \mathbf{A} \right\|^2}, \tag{8}$$

$$\beta_\mu(L) = \frac{\left\| \mathbf{A}^{-1} - \tilde{\mathbf{A}}_\mu^{-1}(L) \right\|^2}{\left\| \mathbf{A}^{-1} \right\|^2}. \tag{9}$$

In its limit, despite the consistency of the initial estimate $\tilde{\mathbf{A}}(L)$, the matrix norm (8) does not reach zero and, other things being equal, will be limited to the value of the fixed regularization parameter μ:

$$\lim_{L\to\infty} \varepsilon_\mu(L) = \mu^2 \frac{N}{\left\| \mathbf{A} \right\|^2} > 0\,. \tag{10}$$

From (10) it follows that for a fixed regularization parameter μ, estimate $\tilde{\mathbf{A}}_\mu(L)$ does not satisfy the criterion of consistency $P\left\{ \lim_{L\to\infty} \varepsilon_\mu(L) = 0 \right\} = 1$.

The calculation of the limit value of the matrix norm $\beta_\mu(L)$ will be carried out based on the spectral decomposition of the asymptotic matrix $\mathbf{A}$, as well as its estimate $\tilde{\mathbf{A}}(L)$ [10,11]:

$$\mathbf{A} = \sum_{i=1}^{N} \lambda_i \mathbf{\Pi}_i\,, \tag{11}$$

$$\tilde{\mathbf{A}}(L) = \sum_{i=1}^{N} \tilde{\lambda}_i(L) \tilde{\mathbf{\Pi}}_i(L)\,, \tag{12}$$

where $\lambda_i = \lambda_i(\mathbf{A})$ and $\tilde{\lambda}_i(L) = \tilde{\lambda}_i\left[\tilde{\mathbf{A}}(L)\right]$ – eigenvalues of matrix $\mathbf{A}$ and its estimate $\tilde{\mathbf{A}}(L)$ respectively; $\mathbf{\Pi}_i = \mathbf{e}_i \mathbf{e}_i^H$ and $\tilde{\mathbf{\Pi}}_i(L) = \tilde{\mathbf{e}}_i(L) \tilde{\mathbf{e}}_i^H(L)$ – projectors of eigenvectors $\mathbf{e}_i = \mathbf{e}_i(\mathbf{A})$ and $\tilde{\mathbf{e}}_i(L) = \tilde{\mathbf{e}}_i\left[\tilde{\mathbf{A}}(L)\right]$ of matrix $\mathbf{A}$ and its estimate $\tilde{\mathbf{A}}(L)$ respectively.

Expressions (11), (12) allow us to represent the asymptotic form of the direct matrix $\mathbf{A}$ as the limit of the spectral decomposition of a consistent estimate $\tilde{\mathbf{A}}(L)$:

$$\mathbf{A} = \sum_{i=1}^{N} \lambda_i \mathbf{\Pi}_i = \lim_{L\to\infty} \sum_{i=1}^{N} \tilde{\lambda}_i(L) \tilde{\mathbf{\Pi}}_i(L)\,. \tag{13}$$

The passage to the limit (13), by virtue of the known lemmas of the theory of limits on the summation of infinitely small and the product of a bounded variable by an infinitely small value, guarantees the existence of the following limits [11]:

$$\lambda_i = \lim_{L\to\infty} \tilde{\lambda}_i(L) \forall i \in [1, N], \tag{14}$$

$$\mathbf{\Pi}_i = \lim_{L\to\infty} \tilde{\mathbf{\Pi}}_i(L)\ \forall i \in [1, N]. \tag{15}$$

In this context, the spectral decomposition of the inverse matrix $\mathbf{A}^{-1}$ and its regularized estimate $\tilde{\mathbf{A}}_\mu^{-1}(L)$ has the following form:

$$\mathbf{A}^{-1} = \sum_{i=1}^{N} \frac{1}{\lambda_i} \mathbf{\Pi}_i, \tag{16}$$

$$\tilde{\mathbf{A}}_\mu^{-1}(L) = \sum_{i=1}^{N} \frac{1}{\tilde{\lambda}_i(L)+\mu} \tilde{\mathbf{\Pi}}_i(L). \tag{17}$$

Based on (16), (17), the value of the matrix norm for an arbitrary sample size L is

$$\beta_\mu(L) = \left(\sum_{i=1}^{N} \frac{1}{\lambda_i^2}\right)^{-1} \times \mathrm{tr}\left\{\sum_{i=1}^{N}\left[\frac{1}{\lambda_i}\mathbf{\Pi}_i - \frac{1}{\tilde{\lambda}_i(L)+\mu}\tilde{\mathbf{\Pi}}_i(L)\right]\right\}^2 .$$

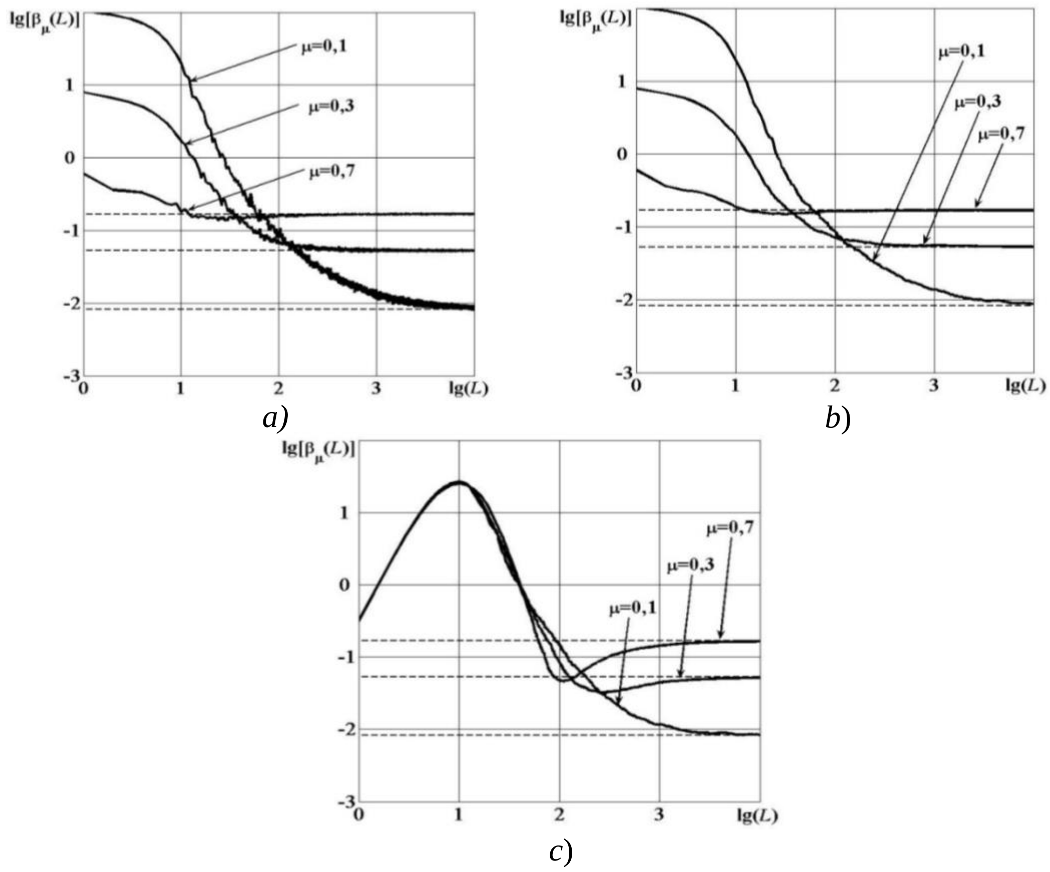


Figure 3 – Dependencies $\beta_\mu(L)$ for a fixed regularization parameter μ obtained according to the different algorithms: (a) – algorithm (3); (b) – algorithm (4); (c) – algorithm (5)

Based on the existence of the limits (14), (15) and condition $\mathrm{tr}\,\mathbf{\Pi}_i = 1$, we have the limit value of the matrix norm $\beta_\mu(L)$:

$$\lim_{L\to\infty}\beta_\mu(L) = \left(\sum_{i=1}^{N}\frac{1}{\lambda_i^2}\right)^{-1}\sum_{i=1}^{N}\left[\frac{\mu}{\lambda_i(\lambda_i+\mu)}\right]^2 > 0\,. \quad (18)$$

Equation (18) demonstrates the inconsistency of regularized estimate $\tilde{\mathbf{A}}_\mu^{-1}(L)$ for a fixed parameter $\mu > 0$: $P\left\{\lim_{L\to\infty}\beta_\mu(L) = 0\right\} \neq 1$.

Figure 3 shows the experimental averaged nonstationary dependencies $\beta_\mu(L)$ that reflect specific properties of the regularized estimates (3)-(5) respectively for fixed values $\mu = 0{,}1$, $\mu = 0{,}3$, and $\mu = 0{,}7$ with the order of the matrix $N = 10$. Here the current values of the matrix norms $\beta_\mu(L)$ are marked with solid curves, and their theoretical limits (18) correspond to the dashed lines.

The approximation of the convergence trajectories $\beta_\mu(L)$ of each of the regularized estimates $\tilde{\mathbf{A}}_\mu^{-1}(L)$ (3)-(5) to the theoretical limit (18) for a finite number of iterations L, shows that all of them are computationally stable but not consistent:

$$\tilde{\mathbf{A}}_\mu^{-1}(L):\begin{cases}\beta_\mu(k) < \infty\ \forall k \in [1, L]\\ P\left\{\lim_{L\to\infty}\beta_\mu(L) = 0\right\} \neq 1\end{cases}.$$

The choice of the value of the static regularization parameter μ is determined by the required accuracy of approximation $\beta_\mu(L)$ for a given sample size L. The selection rule $\mu = \mu[\beta_\mu(L)]$ provides a compromise between the accuracy of approximation $\beta_\mu(L)$ and sample size L. Achieving such a compromise, under conditions of a priori uncertainty about the structure of the spectrum of the correlation matrix $\tilde{\mathbf{A}}_\mu^{-1}(L)$, is problematic. Overcoming this uncertainty can be a variation of parameter μ. At the same time, unjustified variations, such as an increase in the regularization parameter, worsen the correspondence of estimates (3)-(5) to initial data, thereby violating their consistency, and, consequently, the possibility of self-regularization (Figure 3).

Remark 3. There is currently no universal approach to finding the optimal value of the regularization parameter by the criterion "computational stability – consistency" [10-12]. It can be considered a successful approach in which the natural properties of the ML estimates $\tilde{\mathbf{A}}(L)$ and $\tilde{\mathbf{A}}^{-1}(L)$ are used, in particular, their consistency and ability to self-regulate. These properties of estimates $\tilde{\mathbf{A}}(L)$ and $\tilde{\mathbf{A}}^{-1}(L)$ indicate the need to switch from static regularization ($\mu = \text{const}$) to a monotonic decrease of the regularization parameter as the sample size increases: $\lim_{L\to\infty}\mu(L) = 0$. This type of regularization of sample estimates of correlation matrices is classified as dynamic regularization.

DYNAMIC REGULARIZATION OF SAMPLE ESTIMATES OF THE CORRELATION MATRIX

Dynamic regularization of the sample estimate of the correlation matrix is based on the uniqueness theorem for the solution of the inverse problem with perturbed input data [10, 11]. From the theorem, it follows that if the value of the parameter $\mu(L)$, as a monotone function, satisfies condition $\lim_{L\to\infty}\mu(L)=0$ with $\mu(1)>0$, then for regularized estimates $\tilde{\mathbf{A}}_\mu(L)=\tilde{\mathbf{A}}(L)+\mu(L)\mathbf{I}$ and $\tilde{\mathbf{A}}_\mu^{-1}(L)=\left[\tilde{\mathbf{A}}(L)+\mu(L)\mathbf{I}\right]^{-1}$ they converge to the corresponding asymptotic forms:

$$\mathbf{A}=\lim_{L\to\infty}\tilde{\mathbf{A}}_\mu(L),\ \mathbf{A}^{-1}=\lim_{L\to\infty}\tilde{\mathbf{A}}_\mu^{-1}(L).$$

Consequently, in contrast to approximation (18), the matrix norm (9) will have a zero limit $\lim_{L\to\infty}\beta_\mu(L)=0$ and satisfy condition $\beta_\mu(L)<\infty$ for any sample size L. The latter indicates computational stability and consistency of estimate $\tilde{\mathbf{A}}_\mu^{-1}(L)=\left[\tilde{\mathbf{A}}(L)+\mu(L)\mathbf{I}\right]^{-1}$ with the dynamic regularization parameter $\mu(L)$.

In the framework of the method of dynamic regularization, the algorithms of direct (3) and recurrent (4), (5) calculations of the inversion of the matrix estimate $\tilde{\mathbf{A}}_\mu^{-1}(L)$ are transformed to the following form:

$$\mathbf{A}_\mu^{-1}(L)=L\left[\sum_{k=1}^{L}\mathbf{u}_k\mathbf{u}_k^H+\mu(L)\mathbf{I}\right]^{-1}, \tag{19}$$

$$\mathbf{A}_\mu^{-1}(L)\equiv\mathbf{A}_{\mu L}^{-1}=L\left[(L-1)\mathbf{A}_{\mu(L-1)}^{-1}+\mathbf{u}_L\mathbf{u}_L^H+\mu_L\mathbf{I}\right]^{-1}, \tag{20}$$

$$\mathbf{A}_\mu^{-1}(L)\equiv\mathbf{A}_{\mu L}^{-1}=\frac{L}{(L-1)}\left\{\mathbf{I}-\frac{\mathbf{A}_{\mu(L-1)}^{-1}\mathbf{A}_{\mu L}}{L-tr\left[\mathbf{A}_{\mu(L-1)}^{-1}\mathbf{A}_{\mu L}\right]}\right\}\times\mathbf{A}_{\mu(L-1)}^{-1}. \tag{21}$$

The tendency of the behaviour of the convergence trajectories $\beta_\mu(L)$ of estimates (3)-(5) to the asymptotic form $\mathbf{A}$ (Figures 1 and 2) allows, without disturbing the generality of reasoning about the estimates (19)-(21), to restrict ourselves to studying the consistency and computational stability of the algorithm (21). Theoretical studies of the consistency of the ML estimate $\tilde{\mathbf{A}}(L)$ and numerical experiment (Figures 1 and 2) indicate the expediency of using the monotonous law of decrease of the dynamic regularization parameter $\mu(L)$ in practical applications with increasing sample size L, namely the following algorithm:

$$\mu(L)=g(L)L^{-1}. \tag{22}$$

where $g(L)$ – some weight function.

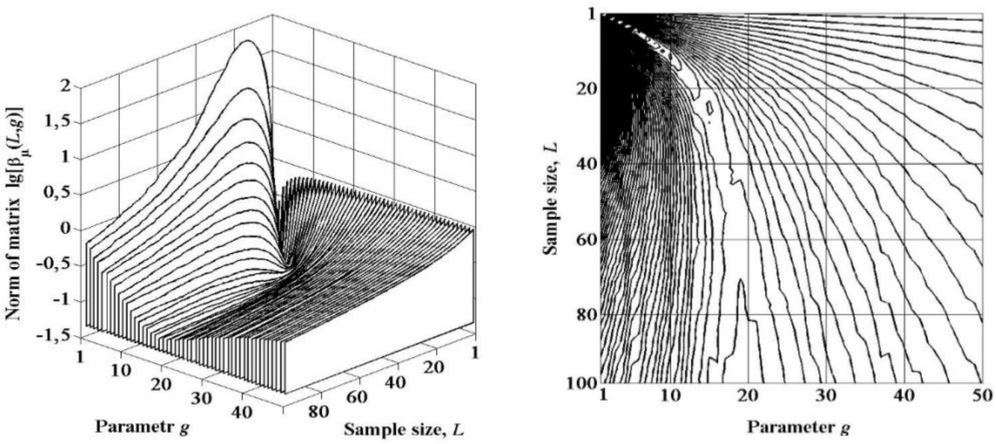
For the algorithm of dynamic regularization (22), the trajectory of convergence $\beta_\mu(L)$ of the matrix estimate $\tilde{\mathbf{A}}_\mu^{-1}(L)$ for arbitrary values $g>0$ of the weight function $g(L)$ generalizes some surface

$$\beta_\mu(L,g)=\frac{\left\|\mathbf{A}^{-1}-\tilde{\mathbf{A}}_\mu^{-1}(L,g)\right\|^2}{\left\|\mathbf{A}^{-1}\right\|^2}.$$

Figure 4 shows a three-dimensional surface $\beta_\mu(L,g)$ (Figure 4,a) and the isolines of the surface $\beta_\mu(L,g)$ (Figure 4,b), obtained with the dimension of the matrix $N=20$.

The analysis of the dependencies shows that due to the quadratic nature of the function $\beta_\mu(L)$ the surface $\beta_\mu(L,g)$ has the so-called "ravine", the coordinates of which satisfy the numerical solution of the optimization problem $\beta_\mu(L)_{opt}$ according to parameter g:

$$\beta_\mu(L)_{opt} = \min_{g\in\Omega_g} \beta_\mu\left(L, g\right),$$



а) б)

Figure 4 – Trajectory of convergence $\beta_\mu(L)$ for arbitrary values of the weight function $g(L)$

where Ω_g – the set of possible values $g > 0$ of the weight function $g(L)$.

In the analogue representation, the coordinates of the trajectory of this "ravine" satisfy a linear inhomogeneous differential equation

$$\frac{\mathrm{d}\,g(t)}{\mathrm{d}t} + \frac{1}{N} g(t) = 1, \;\; g(0) = 1.$$

Solving this equation on a grid of L time samples gives the following result:

$$g(L) = N - (N-1)\exp\left\{-\frac{L-1}{N}\right\}, \;\; g(1) = 1.$$

The resulting expression reflects the process of convergence of the weight function $g(L)$ to its optimal value $g_{opt} = \lim_{L\to\infty} g\left(L\right) = N$. Hence, the algorithm for calculating the optimal parameter of the dynamic regularization $\mu(L)_{opt}$ of the matrix estimate $\tilde{\mathbf{A}}(L)$ has the following form:

$$\mu(L)_{opt} = g_{opt}\, L^{-1} = N\, L^{-1}. \tag{23}$$

The proposed algorithm of dynamic regularization (23) has the following advantages in comparison with the known results [5, 7, 8, 10]:

− uniquely associates the optimal dynamic regularization parameter $\mu(L)_{opt}$ with the dimension N of the correlation matrix and the volume of the observed sample L;

− it is characterized by simplicity of computational operations in real time in the absence of a priori information;

− removes the problem of choosing a regularization parameter under conditions of a priori uncertainty about the initial data of the computational problem.

Remark 4. The decisive advantage of the regularization of $\mu(L)_{opt}$ according to algorithm (23) is the satisfaction of the estimates (19)-(21) of the matrix $\mathbf{A}^{-1}$ of arbitrary dimension N to the criterion "computational stability – consistency". The validity of such an assertion reflects the family of convergence trajectories $\beta_\mu(L)$ of the estimate (21) presented in Figure 5 by solid lines for specific values of $N = 10$, $N = 30$ and $N = 50$ with optimal regularization (23).

Here, for comparison, the dashed lines show the trajectories of convergence $\beta(L)$ of the unregularized estimate (5). These dependencies illustrate the loss of computational stability of the consistent estimate (5) at $L < N$, which is not characteristic of estimate (21) with the optimal dynamic regularization parameter $\mu(L)_{opt}$.

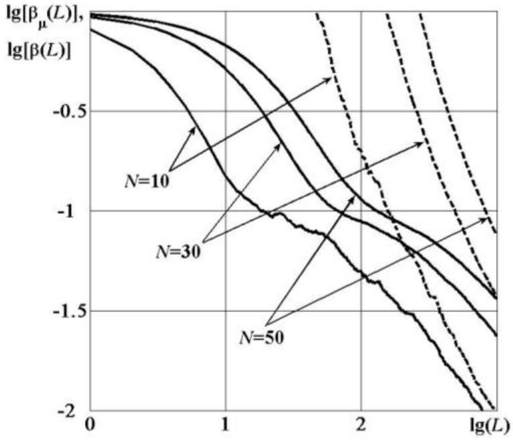


Figure 5 – Convergence trajectories $\beta_\mu(L)$ (solid lines) and $\beta(L)$ (dashed lines)

ILLUSTRATIVE EXAMPLE

Let us consider the application of the dynamic regularization method to solving the problem of detecting an echo-signal at the output of an N-dimensional adaptive antenna array in the conditions of external noise interference. Maximizing the signal-to-noise ratio q at its output involves determining the parametric vector by inverting the estimate of the correlation matrix of observations with the optimal dynamic regularization function

$$\tilde{\mathbf{A}}_\mu^{-1}(L) = \left[\tilde{\mathbf{A}}(L) + \mu(L)_{opt}\,\mathbf{I}\right]^{-1}.$$

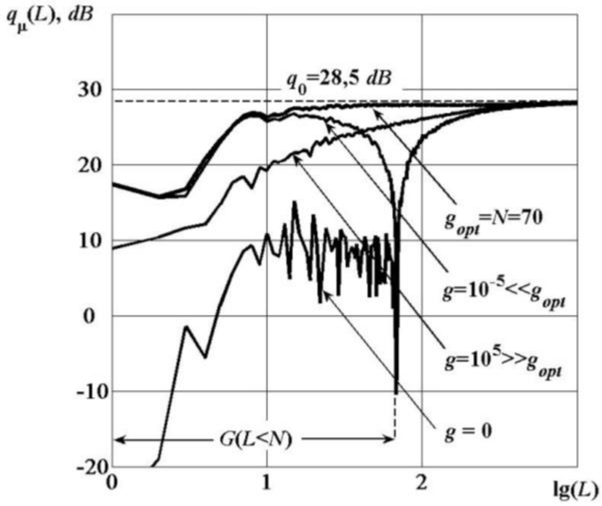


Figure 6 – Transients that illustrate the effect of dynamic regularization $\mu(L) = gL^{-1}$ on the output value of the signal-to-noise ratio $q_\mu(L)$

Figure 6 shows the transients that illustrate the effect of dynamic regularization $\mu(L) = gL^{-1}$ on the output value of the signal-to-noise ratio $q_\mu(L)$ with the dimension of an adaptive antenna array $N = 70$. The presented transients correspond to the conditions of optimal regularization (23), when $g_{opt} = N$; non-optimal regularization, when $g \neq g_{opt}$ ($g = 10^{-5} << g_{opt}$ and $g = 10^5 >> g_{opt}$); self-regularization, when $g = 0$. The potential value of the signal-to-noise ratio is indicated by a dashed line and is $q_0 = 28{,}5\,dB$.

Comparative analysis of dependencies $q_{\mu}(L)$ shows that in the mode of optimal dynamic regularization $\mu(L)_{opt}$, the signal-to-noise ratio achieves its potential value q_0 in virtually a finite number of iterations $L=10$, and the duration of the antenna array adaptation process in comparison with the non-optimal regularization conditions is reduced by at least an order of magnitude.

CONCLUSION

The presented studies address one of the problematic issues of finding optimal solutions to inverse problems under conditions of a priori uncertainty. The optimal in the mean-square approximation algorithm for dynamic regularization of sample estimates of the correlation matrix of observations has been developed. It is proved that the parameter of the weight function should be equal to the dimension of the correlation matrix. The advantages of optimal dynamic regularization in the sense of computational stability and consistency of sample estimates of the correlation matrix of observations are determined. Possessing the property of self-regularization of sample estimates of the correlation matrix, this method represents an alternative to static regularization and allows:

−first, to exclude from the solution of the inverse problem the area of computational instability, in which the information losses are maximum;

−secondly, to obtain a solution to the inverse problem in real time without using forecast data and additional computational costs for finding the optimal value of the regularization parameter.

In terms of practical applications, the optimal dynamic regularization of sample estimates of the correlation matrix of observations expands the capabilities of information systems associated with the solution of incorrect inverse problems under conditions of a priori uncertainty about the initial data.

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